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Supersymmetric IIB solutions with Schrödinger symmetry

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Abstract: We find a class of non-relativistic supersymmetric solutions of IIB supergravity with non-trivial B-field that have dynamical exponent $n = 2$ and are invariant under the Schrödinger group. For a general Sasaki-Einstein internal manifold with $\mathrm{U}(1)^3$ isometry, the solutions have two real supercharges. When the internal manifold is S^5, the number of supercharges can be four. We also find a large class of non-relativistic scale invariant type IIB solutions with dynamical exponents different from two. The explicit solutions and the values of the dynamical exponents are determined by vector eigenfunctions and eigenvalues of the Laplacian on an Einstein manifold.

Keywords: Gauge-gravity correspondence, AdS-CFT Correspondence

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Contents

1 Introduction

The gauge/gravity duality has proven to be a powerful tool to study strongly coupled field theories [1]. There are many strongly coupled condensed matter systems that are of both theoretical and experimental interest. Thus it is reasonable to ask how much can we learn about such field theories using the AdS/CFT correspondence. There has been a lot of effort in this direction as summarized recently in [2, 3].

There are non-relativistic condensed matter systems, like fermions at unitarity, which exhibit the non-relativistic analog of the conformal symmetry - the Schrödinger symmetry. The Schrödinger algebra is generated by spatial translations P^i, temporal translation H, spatial rotations M^{ij}, Galilean boost K^i, the dilatation operator D, a special conformal transformation C and Galilean mass M. In a non-relativistic scale invariant theory, time and space scale differently, $t \to \lambda^n t$ and $\vec{x} \to \lambda\vec{x}$, respectively. The real parameter n is called the dynamical exponent and the Schrödinger invariant systems have $n = 2$. When $n \neq 2$, one does not have the special conformal generator in the algebra and the theory is only scale invariant. The familiar case of scale invariance in a relativistic conformal theory corresponds to $n = 1$. More details on non-relativistic conformal theories can be found in [4–6].

In [7, 8], a five-dimensional gravitational background with Schrödinger symmetry was found as a solution to the Einstein-Hilbert action coupled to a massive vector field. Subsequently in [9–11] this solution was embedded in the type IIB supergravity.[1] The ten-dimensional background was obtained by applying a solution generating technique, known as the null Melvin twist, to the $AdS_5 \times S^5$ background. The null Melvin twist, described in [24, 25], can be used to generate new supergravity solutions starting from a known solution, when the latter has at least one compact and one noncompact isometry. This technique has been extensively used in, e.g., [16, 22] to generate new non-relativistic gravity backgrounds.

There are also supersymmetric extensions of the Schrödinger algebra, which have been studied in [26–31]. Therefore it is natural to look for Schrödinger invariant supergravity solutions which possess some supersymmetry. The Schrödinger invariant solutions discussed in [9–11] completely break supersymmetry even though they are obtained from supersymmetric IIB backgrounds. A supersymmetric Schrödinger invariant solution was constructed in [15], however, this solution is unstable, has vanishing B-field and is sourced only by the usual self-dual RR flux. Other examples of supersymmetric non-relativistic solutions were found in [15, 19, 21, 23], however, all those solutions have dynamical exponent $n \neq 2$ and therefore are scale invariant but not invariant under the full Schrödinger group.

In this paper we will analyze in detail the supersymmetries preserved by non-relativistic Schrödinger invariant solutions of the type IIB supergravity with non-vanishing B-field. We consider non-relativistic backgrounds generated by the null Melvin twist applied to the Freund-Rubin type solutions of the form $AdS_5 \times X_5$, where X_5 is a Sasaki-Einstein manifold. When the Killing spinor of the Sasaki-Einstein manifold is invariant under the U(1) isometry used in the twist, the non-relativistic solution preserves, in general, two real supercharges. These two supercharges are a subset of the Poincaré supersymmetries of the relativistic superconformal algebra. The superconformal symmetries are completely broken by the twist.

We illustrate our general result with two familiar examples: S^5 and $T^{1,1}$. In the first case, we find that the non-relativistic solution can have four real supersymmetries. This is due to the 32 unbroken supersymmetries in the original solution on S^5 before the twist. The second case illustrates better the generic situation where only 8 supersymmetry are present before the twist. Other examples that are covered by our analysis are generalizations of the $T^{1,1}$ example and include two infinite families of Sasaki-Einstein manifolds, $Y^{p,q}$ [32] and $L^{p,q,r}$ [33]. All those spaces can be used as internal manifolds for supersymmetric Schrödinger invariant IIB solutions. Since both of these infinite families have U(1) isometries that leave the Killing spinor invariant, we find an infinite number of Schrödinger invariant solutions which preserve two supercharges.

The general form of the backgrounds constructed by the null Melvin twist also suggests a natural Ansatz for non-relativistic type IIB solutions with higher dynamical exponents and non-zero B-field. We show that there is a large class of such solutions of the form

$$\begin{aligned} ds^2_{10} &= -\frac{\Omega}{z^{2n}}du^2 + \frac{1}{z^2}(-2\,dudv + dx_1^2 + dx_2^2 + dz^2) + ds^2_{X_5}\,, \\ F_{(5)} &= (1+\star)\,\mathrm{vol}_{X_5}\,, \qquad\qquad B_{(2)} = \frac{1}{z^n}\mathcal{A}\wedge du\,, \end{aligned}$$

[1] See [12]–[23] for examples of other non-relativistic gravity solutions.

where X_5 is an Einstein manifold and $\mathcal{A}$ is an one-form on X_5. We find that $\mathcal{A}$ must be a vector eigenfunction of the Laplacian on X_5 and the dynamical exponent, n, is determined by the corresponding eigenvalue. The metric function Ω obeys an inhomogeneous scalar Laplace equation on X_5. In principle, both $\mathcal{A}$ and Ω can be determined explicitly using harmonic expansions.

The class of solutions constructed here includes all solutions generated by the null Melvin twist and also the solutions with general dynamical exponents and vanishing B-field found in [15]. It is worth emphasizing that in the more general case of solutions with a nontrivial B-field, the dynamical exponent is related to the eigenvalues of vector harmonics on Einstein manifolds.

The paper is organized as follows: In section 2, we present the non-relativistic Schrödinger invariant supergravity backgrounds obtained by the null Melvin twist and recast them in a form that is convenient for analysis of unbroken supersymmetries carried out in detail in section 3. Then, in section 4, we work out some explicit examples that illustrate the general discussion in section 3. In section 5, we introduce an Ansatz for type IIB solutions with general dynamical exponents and show that it reduces to a coupled system of a vector and a scalar Laplace equations on the internal manifold. We also work out in detail some examples on S^5 using standard methods of harmonic expansion. We conclude in section 6 with comments and directions for further study. A brief discussion of the null Melvin twist and a summary of some pertinent solutions are given in the appendix.

2 The solution

Consider a Freund-Rubin type solution of IIB supergravity of the form $AdS_5 \times X_5$ with the metric and the five-form flux given by

$$ds^2_{10} = ds^2_{AdS_5} + ds^2_{X_5} \,, \tag{2.1}$$

$$F_{(5)} = (1+\star)\mathrm{vol}_{AdS_5} \,, \tag{2.2}$$

where X_5 is an Einstein manifold. In addition we assume that X_5 has at least U(1) isometry, with the corresponding Killing vector $\mathcal{K}$.

In the following we will use the metric on AdS_5 written in terms of light-cone coordinates,

$$ds^2_{AdS_5} = \frac{1}{z^2}(-2\,du dv + dx_1^2 + dx_2^2 + dz^2) \,, \tag{2.3}$$

with the radius of AdS_5 normalized to one.

The null Melvin twist [24, 25] along a Killing vector $\mathcal{K}$ on X_5 yields another type IIB solution of the form $Sch_5 \times X_5$, where Sch_5 is a five-dimensional space-time invariant under the Schrödinger symmetry.[2] The metric,

$$ds^2_{10} = ds^2_{Sch_5} + ds^2_{X_5} \,, \tag{2.4}$$

[2]See the appendix for more details of this construction.

the five-form flux

$$F_{(5)} \;=\; (1+\star)\mathrm{vol}_{Sch_5}\,, \tag{2.5}$$

and, in addition, a nonzero three-form flux, $H_{(3)} = dB_{(2)}$, in the solution can be written explicitly in terms of the data of the initial solution (2.1), (2.2) and $\mathcal{K}$. Specifically, the metric along Sch_5 in the same light-cone coordinates as above is

$$ds^2_{Sch_5} \;=\; -\,\frac{\Omega}{z^4}du^2 + \frac{1}{z^2}(-2\,dudv + dx_1^2 + dx_2^2 + dz^2)\,, \tag{2.6}$$

and

$$\Omega = ||\mathcal{K}||^2\,, \tag{2.7}$$

is a nonnegative function given by the length square of the Killing vector, $\mathcal{K}$, with respect to the metric on X_5. Similarly, the two-form potential is given by

$$B_{(2)} \;=\; \frac{1}{z^2}\mathcal{K}\wedge du\,, \tag{2.8}$$

where $\mathcal{K}$ is the one-form dual to $\mathcal{K}$.[3] To make sense of these solutions as holographic duals to non-relativistic field theories the light-cone coordinate v should be periodically identified $v \sim v + 2\pi r_v$ [7, 8, 11]. The momentum along this compact direction is quantized in units of the inverse radius r_v^{-1}. This momentum is interpreted as the Galilean mass (or the particle number) in the dual field theory.

By construction, (2.4), (2.5) and (2.8) satisfy the equations of motion[4] of IIB supergravity for any Killing vector $\mathcal{K}$, since all one is using is a series of boosts, T-dualities and shifts which are all symmetries of IIB supergravity. One can also view the $Sch_5 \times X_5$ solution as a deformation of the $AdS_5 \times X_5$ solution above, which can be formally recovered by setting $\mathcal{K} = 0$ in (2.4), (2.5) and (2.8).

It is possible that the norm of the Killing vector $\mathcal{K}$ vanishes on some locus in X_5. The curvature of the solution is completely regular on this locus, in fact, the solution looks like $AdS_5 \times X_5$. It is somewhat strange that the asymptotic structure of the non-compact space changes from the non-relativistic Sch_5 to AdS_5 at special points on X_5. However this kind of space-times have been analyzed in the literature, see [25, 34] and references therein. One can argue that, despite the presence of the locus on which Ω vanishes, the ten-dimensional background is non-distinguishing and thus has the proper asymptotic and causal structure for a dual of a non-relativistic field theory. An intuitive way to understand this is to observe that in the solution (2.4) every point with $u > u_0$ can be reached by a causal curve on the ten-dimensional background starting at u_0. This implies that the light-cone is degenerate and the space-time is non-distinguishing. This is precisely a property one should expect from a gravity dual to a non-relativistic field theory. The presence of the locus on which Ω vanishes does not change the fact that the space-time is non-distinguishing as long as Ω is non-zero on an open set in X_5. This will always be the case for our solutions.[5]

[3] In terms of explicit coordinates ξ^α on X_5, we have $ds^2_{X_5} = g_{\alpha\beta}d\xi^\alpha d\xi^\beta$, $\Omega = g_{\alpha\beta}\mathcal{K}^\alpha\mathcal{K}^\beta$, and $\mathcal{K}_\alpha = g_{\alpha\beta}\mathcal{K}^\beta$.
[4] We have also checked this explicitly, see the appendix for more details.
[5] We are grateful to Veronika Hubeny and Mukund Rangamani for helpful explanations on this point.

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3 General supersymmetry analysis

Let us now assume that the $AdS_5 \times X_5$ solution preserves some of the supersymmetries of IIB supergravity, that is there exists a chiral Killing spinor ϵ_0 in ten dimensions,

$$\Gamma^1 \ldots \Gamma^{10} \epsilon_0 \;=\; \epsilon_0\,, \tag{3.1}$$

for which the dilatino and the gravitino supersymmetry variations vanish. In the following we will find sufficient conditions under which the Killing spinor, ϵ_0, can be deformed to a Killing spinor, ϵ, of the $Sch_5 \times X_5$ solution.

To this end let us introduce the frames, e^M, for the metric (2.4),

$$\begin{aligned} e^1 &= \frac{1}{2z^2}(\Omega+1)\,du + dv\,, & e^4 &= \frac{1}{2z^2}(\Omega-1)\,du + dv\,, & & \\ e^2 &= \frac{1}{z}dx_1\,, & e^3 &= \frac{1}{z}dx_2\,, & e^5 &= \frac{1}{z}dz\,, \\ & & e^{5+\alpha} &= e^{\alpha}_{(5)}\,, \quad \alpha = 1,\ldots,5\,. & & \end{aligned} \tag{3.2}$$

where $e^{\alpha}_{(5)}$ are some orthonormal frames on X_5 that will be specified later. The equations for unbroken supersymmetry are [35]

$$\delta\lambda \;=\; -\frac{1}{24} H_{MNP}\Gamma^{MNP}\epsilon^* \;=\; 0\,, \tag{3.3}$$

and

$$\delta\psi_M \;=\; \nabla_M \epsilon + \frac{i}{480} F_{NPQRS}\Gamma^{NPQRS}\Gamma_M\,\epsilon - \frac{1}{48}\left[\Gamma_M, H_{PQR}\Gamma^{PQR}\right]\,\epsilon^*\,, \tag{3.4}$$

where the flat indices $M,N,\ldots$ range from 1 to 10. We use the same conventions as in [36] with the mostly plus metric and real Dirac Γ-matrices in ten dimensions.

The corresponding frames, e_0^M, and the supersymmetry equations, $\delta_0\lambda = 0$ and $\delta_0\psi_M = 0$ for the $AdS_5 \times X_5$ solution are obtained by setting $\mathcal{K} = 0$ in (3.2), (3.3) and (3.4). In that case the $H_{(3)}$ flux vanishes and the dilatino variation vanishes identically.

We start our analysis of unbroken supersymmetries with the dilatino variation (3.3) in which the $H_{(3)}$ flux is given by

$$H_{(3)} \;=\; dB_{(2)} \;=\; \left(d\mathcal{K} + 2\mathcal{K}\wedge e^5\right)\wedge(e^1 - e^4)\,. \tag{3.5}$$

Since $\mathcal{K}$ is a one form on X_5, it follows that (3.3) factorizes into

$$\delta\lambda \;=\; -\frac{1}{24} H_{MNP}\Gamma^{MNP}\epsilon^* \;=\; \mathbb{M}(\Gamma^1 - \Gamma^4)\epsilon^* \;=\; 0\,, \tag{3.6}$$

where $\mathbb{M}$ is a real matrix

$$\mathbb{M} \;=\; -\frac{1}{8}(d\mathcal{K})_{MN}\Gamma^{MN} + \frac{1}{2}\mathcal{K}_M\Gamma^{5M}\,. \tag{3.7}$$

Note that the summation above is over the range $M,N = 6,\ldots,10$ since both $\mathcal{K}$ and $d\mathcal{K}$ have nonvanishing components only along X_5. Hence we can solve (3.6) by imposing a single projection condition

$$(\Gamma^1 - \Gamma^4)\epsilon^* \;=\; (\Gamma^1 - \Gamma^4)\epsilon \;=\; 0\,, \tag{3.8}$$

where the condition for ϵ follows using reality of the Γ-matrices.

The gravitino variations (3.4) involve two types of terms that depend on $\mathcal{K}$ and thus are absent in the corresponding equations for the Kiling spinor ϵ_0 on $AdS_5 \times X_5$. If those terms can be eliminated from the equations, the problem of finding the Killing spinor ϵ on $Sch_5 \times X_5$ will be reduced to that of finding ϵ_0 on $AdS_5 \times X_5$.

The additional terms of the first type arise from the deformation of the spin connection due to the function Ω in e^1 and e^4. We can write this deformation succinctly as the difference of the spin connections for the two metrics,

$$W - W\big|_{\Omega=0} \;=\; \frac{4\Omega}{z^5}\, du \otimes du \wedge dz - \frac{1}{z^4}\, du \otimes du \wedge d\Omega\,, \tag{3.9}$$

where

$$W \;=\; \omega_{MN} \otimes e^M \wedge e^N\,, \tag{3.10}$$

and ω_{MN} is the spin connection. It is clear that the deformation due to those additional terms will arise only in the $\delta\psi_u$ variation.[6]

The deformation terms involving the $H_{(3)}$ flux manifestly vanish due to (3.8) for all $\delta\psi_M$, but $\delta\psi_1$ and $\delta\psi_4$. Indeed, for $M \neq 1, 4$, the $(\Gamma^1 - \Gamma^4)$ factor arising from the contraction as in (3.6) commutes, or anticommutes, with all other matrices in this term. Hence it can be moved to act directly on ϵ^*, so that these variations vanish due to (3.8).

To evaluate the remaining two variations, consider the combination $e^1\delta\psi_1 + e^4\delta\psi_4$. This yields a sum of two terms

$$\frac{1}{96}\left(\frac{\Omega}{z^2}du + 2\,dv\right)\left[\Gamma^1 - \Gamma^4, H_{MNP}\Gamma^{MNP}\right]\epsilon^* + \frac{1}{96z^2}du\left[\Gamma^1 + \Gamma^4, H_{MNP}\Gamma^{MNP}\right]\epsilon^*\,. \tag{3.11}$$

The first commutator vanishes identically, while the second one gives a nontrivial contribution to $\delta\psi_u$, which we evaluate explicitly below.

To summarize, we have shown that, apart from the dilatino variation, the only equation that is modified by the deformation is the gravitino variation $\delta\psi_u$. Before we proceed with this variation, let us note that the other gravitino variations along Sch_5 are solved by a single additional projector,

$$\left(\Gamma^2 + i\Gamma^3\right)\epsilon \;=\; 0\,. \tag{3.12}$$

Indeed, upon using (3.1) to simplify the $F_{(5)}$ flux terms, and then imposing the projection (3.8), all variations $\delta\psi_{x_1}$, $\delta\psi_{x_2}$, $\delta\psi_v$ and $\delta\psi_z$ reduce to (3.12) multiplied by some other Γ-matrix.

Note that in the case of the $AdS_5 \times X_5$ solution, the gravitino variations along AdS_5 are solved by a single projector

$$\left(1 - i\Gamma^{1234}\right)\epsilon_0 \;=\; 0\,. \tag{3.13}$$

The solutions to this equation include both solutions to (3.8) and (3.12) and to the equations where both projectors are replaced by the ones with the opposite sign.

[6] We use a shorthand notation for the curved indices labelling them with the corresponding coordinate.

Finally, consider the variation $\delta\psi_u$. Here we find

$$\begin{aligned}\delta\psi_u - \delta_0\psi_u \;&=\; -\frac{1}{4z^2}\left(3\,\Omega\,\Gamma^5 - \Gamma^\alpha\partial_\alpha\Omega\right)(\Gamma^1-\Gamma^4)\,\epsilon \\ &\quad + \frac{i}{4z^2}\,\Omega\,\Gamma^2\Gamma^3\Gamma^5(\Gamma^1-\Gamma^4)\,\epsilon \\ &\quad - \frac{1}{z^2}\,\mathbb{M}\,\epsilon^*\,. \end{aligned} \tag{3.14}$$

The terms in the first line are due to (3.9), and we have introduced a shorthand notation $\Gamma^\alpha = e^\alpha_M\Gamma^M$. The second line arises from additional terms in the $F_{(5)}$ flux in the coordinate basis due to the Ω-terms in e^1 and e^4. The last line is due to the non-vanishing term in (3.11) with the matrix $\mathbb{M}$ given in (3.7). Clearly the first two lines vanish if we impose (3.8), which leaves a single additional algebraic constraint on the Killing spinor,

$$\mathbb{M}\,\epsilon^* \;=\; 0\,. \tag{3.15}$$

In the following we will unravel the conditions under which this equation has nontrivial solutions.

The transformation of a Killing spinor, ϵ, under an isometry $\mathcal{K}$ is given by the Lie derivative

$$\begin{aligned}\mathcal{L}_\mathcal{K}\,\epsilon \;&=\; \mathcal{K}^M\partial_M\epsilon + \frac{1}{4}\left(\mathcal{K}^M\omega_{MPQ} + \nabla_{[P}\mathcal{K}_{Q]}\right)\Gamma^{PQ}\epsilon \\ &=\; \mathcal{K}^M\nabla_M\epsilon + \frac{1}{8}(d\mathcal{K})_{MN}\Gamma^{MN}\epsilon\,. \end{aligned} \tag{3.16}$$

Next consider the gravitino variation along $\mathcal{K}$,

$$\mathcal{K}^M\delta\psi_M \;=\; \mathcal{K}^M\nabla_M\epsilon + \frac{i}{480}F_{NPQRS}\Gamma^{NPQRS}(\mathcal{K}^M\Gamma_M)\,\epsilon\,. \tag{3.17}$$

The second term can be expanded using the explicit form of the $F_{(5)}$ flux in (2.5). We get, using (3.1) and (3.13),

$$\frac{i}{480}F_{NPQRS}\Gamma^{NPQRS}(\mathcal{K}^M\Gamma_M)\,\epsilon \;=\; \frac{1}{2}\mathcal{K}_M\Gamma^{5M}\epsilon\,.$$

Substituting this back in (3.17) and using (3.16) and (3.7), we obtain

$$\begin{aligned}\mathcal{K}^M\delta\psi_M \;&=\; \mathcal{L}_\mathcal{K}\,\epsilon - \frac{1}{8}(d\mathcal{K})_{MN}\Gamma^{MN}\epsilon + \frac{1}{2}\mathcal{K}_M\Gamma^{5M}\epsilon \\ &=\; \mathcal{L}_\mathcal{K}\,\epsilon + \mathbb{M}\,\epsilon\,. \end{aligned} \tag{3.18}$$

This shows that a Killing spinor, ϵ, is annihilated by $\mathbb{M}$ if and only if it is invariant under the corresponding isometry. Since $\mathbb{M}$ is real, this also implies that $\mathbb{M}\,\epsilon^* = 0$.

The result of our analysis is an explicit method for obtaining Killing spinors for the Schrödinger background, $Sch_5 \times X_5$ obtained by the null Melvin twist, starting with the Killing spinors of the undeformed $AdS_5 \times X_5$ background:

- *A Killing spinor, ϵ, on $AdS_5 \times X_5$ is also a Killing spinor on $Sch_5 \times X_5$, where the $Sch_5 \times X_5$ solution is obtained by the null Melvin twist along the Killing vector $\mathcal{K}$, provided ϵ satisfies*

$$\frac{1}{2}\left(1+\Gamma^{14}\right)\epsilon \;=\; \frac{1}{2}\left(1+i\,\Gamma^{23}\right)\epsilon \;=\; 0 \qquad \text{and} \qquad \mathcal{L}_{\mathcal{K}}\,\epsilon \;=\; 0\,.$$

- *Conversely, any Killing spinor on $Sch_5 \times X_5$ satisfying the projections*[7] *above gives rise to a $\mathcal{K}$-invariant Killing spinor on $AdS_5 \times X_5$.*

In fact, it appears that the above construction gives rise to *all* Killing spinors on $Sch_5 \times X_5$. The complete analysis is more involved. If we start with an ϵ that does not satisfy (3.8), we must solve the dilatino variation by setting $\mathbb{M}\,\epsilon^* = 0$ from the start. Furthermore, in all gravitino variations, the $H_{(3)}$ flux terms will not cancel. A systematic method to exclude this type of Killing spinors would be to analyze the integrability conditions for the gravitino variations. We have not carried out this calculation in the general case, but rather verified explicitly in the simplest examples of $X_5 = S^5$ and $X_5 = T^{1,1}$ that there are no further Killing spinors of opposite Γ^{14} chirality. This is in agreement with [15, 19], where it was shown that non-relativistic supersymmetric solutions with vanishing $H_{(3)}$ flux and different dynamical exponents break all superconformal Killing supersymmetries.

The undeformed $AdS_5 \times X_5$ solution has $4 \otimes 2 = 8$ real supercharges where the factor of 4 in the direct product comes from the AdS_5 Killing spinors and 2 is the number of Killing spinors on a generic Sasaki-Einstein manifold. As discussed above, the null Melvin twist breaks all AdS_5 supersymmetries and we are left with non-relativistic solutions preserving 2 supercharges. For the case of $AdS_5 \times S^5$ we have $4 \otimes 8 = 32$ supersymmetries because S^5 has 8 Killings spinors and as we discuss in the next section one can find cases in which the number of supersymmetries of the twisted solution is enhanced to 4.

4 Examples

In this section we illustrate how the general construction in the previous section works for some well known five-dimensional Sasaki-Einstein manifolds.

4.1 S^5

The most symmetric example of a five-dimensional Sasaki-Einstein manifold is the sphere S^5. It has SO(6) isometry group and we can apply the general, three-parameter null Melvin twist on a $\mathrm{U}(1)^3$ subgroup. One can find the conditions for unbroken supersymmetry and construct Killing spinors of the twisted solution explicitly, however, it is more efficient to use group theory to extract this information.

The SO(6) isometry group of S^5 is generated by the Killing vectors $M_{IJ} = x_I \partial_J - x_J \partial_I$, where we realize S^5 as a unit sphere $x_1^2 + \ldots + x_6^2 = 1$ in $\mathbb{R}^6$. If we choose the $\mathrm{U}(1)^3$ Cartan subalgebra generators as

$$\mathcal{K}_{(1)} \;=\; M_{12}\,, \qquad \mathcal{K}_{(2)} \;=\; M_{34}\,, \qquad \mathcal{K}_{(3)} \;=\; M_{56}\,, \tag{4.1}$$

[7] As was noted above, those two projections corresponding to (16) and (20) are related by the projection (21) on $AdS_5 \times X_5$.

we find that for a Killing vector $\mathcal{K} = \eta_1\mathcal{K}_1 + \eta_2\mathcal{K}_2 + \eta_3\mathcal{K}_3$,

$$\Omega \;=\; ||\mathcal{K}||^2 \;=\; \eta_1^2(x_1^2+x_2^2) + \eta_2^2(x_3^2+x_4^2) + \eta_3^2(x_5^2+x_6^2)\,. \tag{4.2}$$

The Killing spinors on S^5 transform in $\mathbf{4}\oplus\bar{\mathbf{4}}$ of SO(6). Their charges with respect to the $\mathrm{U}(1)^3$ above are

$$\mathbf{4} \quad\longrightarrow\quad (+1,+1,+1)\oplus(+1,-1,-1)\oplus(-1,+1,-1)\oplus(-1,-1,+1)\,. \tag{4.3}$$

Hence the Killing spinors invariant under $\mathcal{K}$ are determined by solutions to the equation

$$\eta_1 \pm \eta_2 \pm \eta_3 \;=\; 0\,. \tag{4.4}$$

For values of η_i satisfying (4.4), the non-relativistic solution preserves two real supersymmetries. If in addition to (4.4) we impose that at least one of the η_i vanishes, the number of real supercharges is enhanced to four. One can verify explicitly that all sixteen superconformal Killing spinors of $AdS_5\times S^5$ are broken by the null Melvin twist so that one can preserve only a subset of the Poincaré Killing spinors. Of course when all η_i vanish we get back to the original $AdS_5\times S^5$ background, which preserves sixteen superconformal and sixteen Poincaré supersymmetries.

It is clear that Ω is strictly positive when none of the η_i's vanish. Setting one η_i to zero, say $\eta_3=0$, the Killing vector $\mathcal{K}$ vanishes on S^1 given by $x_1=x_2=x_3=x_4=0$ and $x_5^2+x_6^2=1$. Similarly, when two η_i vanish, there is an S^3 on which $\mathcal{K}$ vanishes. As we discussed in section 2, even though there could be a locus on which Ω vanishes, the twisted background is still non-distinguishing and thus non-relativistic.

The special case $\eta_1=\eta_2=\eta_3=\eta$ corresponds to the null Melvin twist along the Hopf fiber of S^5 and has been studied in [9–11]. In this case the Killing vector $\mathcal{K}$ has constant norm and it is clear from (4.4) that the twisted solution breaks supersymmetry completely. This has been also shown explicitly in [10].

4.2 $T^{1,1}$

In this section we describe explicitly the null Melvin twists for the $AdS_5\times T^{1,1}$ solution [37]. Recall that $T^{1,1}$ is the coset space $\mathrm{SU}(2)\times\mathrm{SU}(2)/\mathrm{U}(1)$ with a unique homogenous Einstein metric. There is a Killing spinor on $T^{1,1}$, with two real components, which gives rise to the $\mathcal{N}=1$ unbroken supersymmetry of the Romans solution. The isometries of the solution arise from the obvious $\mathrm{SU}(2)\times\mathrm{SU}(2)$ action on the left and, in addition, from another U(1) action from the right on the coset. The Killing spinor is necessarily invariant under $\mathrm{SU}(2)\times\mathrm{SU}(2)$, and transforms nontrivially under the U(1), which is the R symmetry of the $\mathcal{N}=1$ superalgebra.

It is convenient to realize $T^{1,1}$ explicitly as a locus in $\mathbb{C}^4$ by introducing a complex matrix [38]

$$\mathcal{W} \;=\; \frac{1}{\sqrt{2}}\begin{pmatrix} z_3+iz_4 & z_1-iz_2\\ z_1+iz_2 & -z_3+iz_4\end{pmatrix}\,, \tag{4.5}$$

subject to the constraints

$$\mathrm{Tr}\,\mathcal{W}^\dagger\mathcal{W}=1\,,\qquad \det\mathcal{W} \;=\; 0\,. \tag{4.6}$$

In this parametrization, the two SU(2)'s, call them $\mathrm{SU}(2)_1$ and $\mathrm{SU}(2)_2$, act on $\mathcal{W}$ by left and right multiplication, respectively, while the R-symmetry, $\mathrm{U}(1)_R$, corresponds to the phase rotation, $z_i \to e^{i\phi_3} z_i$. As explained in [38], the constraints (4.5) can be solved explicitly by introducing the Euler angles, $(\theta_1, \phi_1, \psi_1)$ and $(\theta_2, \phi_2, \psi_2)$ for $\mathrm{SU}(2)_1$ and $\mathrm{SU}(2)_2$, and setting $\psi_1 = \psi_2 = \phi_3/2$ to pass onto the coset. In terms of those angles, the unique Einstein metric on $T^{1,1}$ is [38]

$$ds^2_{T^{1,1}} = \frac{1}{6}(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2 + d\theta_2^2 + \sin^2\theta_2 d\phi_2^2) + \frac{1}{9}(d\phi_3 + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2 \,. \quad (4.7)$$

For the general null Melvin twist we choose Killing vectors $\mathcal{K}_{(1)}$, $\mathcal{K}_{(2)}$ and $\mathcal{K}_{(3)}$ corresponding to $\mathrm{U}(1)_1 \subset \mathrm{SU}(2)_1$, $\mathrm{U}(1)_2 \subset \mathrm{SU}(2)_2$ and $\mathrm{U}(1)_R$, normalized such that

$$\mathcal{K}_{(1)} = \frac{\partial}{\partial\phi_1}\,, \qquad \mathcal{K}_{(2)} = \frac{\partial}{\partial\phi_2}\,, \qquad \mathcal{K}_{(3)} = \frac{\partial}{\partial\phi_3}\,. \quad (4.8)$$

One can perform the null Melvin twist along ϕ_i and the Killing vector defining the non-relativistic solution is

$$\mathcal{K} = \sum_{i=1}^{3} \eta_i \frac{\partial}{\partial\phi_i}\,. \quad (4.9)$$

The function Ω is

$$\Omega = \eta_1^2\left(\frac{\sin^2\theta_1}{6} + \frac{\cos^2\theta_1}{9}\right) + \eta_2^2\left(\frac{\sin^2\theta_2}{6} + \frac{\cos^2\theta_2}{9}\right) + \frac{\eta_3^2}{9} + 2\eta_1\eta_2\frac{\cos\theta_1\cos\theta_2}{9} + 2\eta_1\eta_3\frac{\cos\theta_1}{9} + 2\eta_2\eta_3\frac{\cos\theta_2}{9}\,. \quad (4.10)$$

The matrix $\mathbb{M}$ is given by

$$\mathbb{M} = \frac{\eta_1}{2}\left(\frac{\sin\theta_1}{\sqrt{6}}(\Gamma^{58} + \Gamma^{610}) - \frac{\cos\theta_1}{3}(2\Gamma^{68} - \Gamma^{79} - \Gamma^{510})\right) + \frac{\eta_2}{2}\left(\frac{\sin\theta_2}{\sqrt{6}}(\Gamma^{59} + \Gamma^{710}) - \frac{\cos\theta_2}{3}(2\Gamma^{79} - \Gamma^{68} - \Gamma^{510})\right) + \frac{\eta_3}{6}(\Gamma^{68} + \Gamma^{79} + \Gamma^{510})\,. \quad (4.11)$$

One can show that the Killing spinor for this solution is

$$\epsilon = e^{-\frac{i}{2}\phi_3}\widetilde{\epsilon}_0\,, \quad (4.12)$$

where $\widetilde{\epsilon}_0$ is a constant spinor satisfying the chirality condition (3.1) and four additional projectors

$$(1 + \Gamma^{14})\widetilde{\epsilon}_0 = (1 + i\Gamma^{23})\widetilde{\epsilon}_0 = (1 + i\Gamma^{68})\widetilde{\epsilon}_0 = (1 + i\Gamma^{79})\widetilde{\epsilon}_0 = 0\,. \quad (4.13)$$

The condition $\mathbb{M}\,\epsilon^* = 0$ is satisfied by (4.12) if $\eta_3 = 0$.

Thus we find that the generalized null Melvin twist of $AdS_5 \times T^{1,1}$ with the parameters $(\eta_1, \eta_2, 0)$ for non-zero (η_1, η_2) leads to a type IIB solution of the form $Sch_5 \times T^{1,1}$ with $H_{(3)}$ and $F_{(5)}$ flux, which preserves two real supercharges. For $\eta_3 \neq 0$, we still have a Schrödinger invariant type IIB solution, but the supersymmetry is completely broken.

4.3 $Y^{p,q}$ and $L^{p,q,r}$

There are two infinite families of five-dimensional Sasaki-Einstein manifolds with explicitly known metrics. The manifolds in the $Y^{p,q}$ family, found in [32], are specified by two integers (p,q) determined by some regularity conditions.[8] The solutions have $\mathrm{SU}(2) \times \mathrm{U}(1) \times \mathrm{U}(1)_R$ symmetry. The $L^{p,q,r}$ solutions are specified by a set of integers (p,q,r) and have even smaller isometry group, $\mathrm{U}(1)^2 \times \mathrm{U}(1)_R$ [33]. The $\mathrm{U}(1)_R$ isometry is special and the Killing vector[9] corresponding to it has a constant norm. Such a Killing vector exists on every Sasaki-Einstein manifold, X_5, and it can be determined by the Kähler form on the corresponding Calabi-Yau cone over X_5 [39]. The Killing spinor on X_5 has two real components and is charged under $\mathrm{U}(1)_R$ so the results of section 3 imply that the null Melvin twist along $\mathrm{U}(1)_R$ will break supersymmetry completely. However, for both the $Y^{p,q}$ and the $L^{p,q,r}$ families we have two additional $\mathrm{U}(1)$ isometries along which we can apply the twist with arbitrary real parameters (η_1, η_2). The resulting non-relativistic solutions will be Schrödinger invariant and will preserve two real supercharges that are a subset of the Poincaré supersymmetries of $AdS_5 \times X_5$. The superconformal charges are completely broken by the twist. Since the metrics on both $Y^{p,q}$ and $L^{p,q,r}$ are explicitly known, one can in principle construct explicit Killing spinors on them.

5 New solutions from vector harmonics

In this section we introduce a new class of solutions with Galilean symmetry, general dynamical exponents and nontrivial three-form flux that are generated by vector harmonics on X_5. The starting point of our construction is an Ansatz that is a natural generalization of the twisted solutions in sections 2 and 3 and the solutions with general dynamical exponents, but without $H_{(3)}$ flux, constructed recently by Hartnoll and Yoshida [15] using scalar harmonics on X_5.

In the notation of section 2, the metric in [15] is of the form

$$ds^2 \;=\; -\frac{\Omega}{z^{2n_1}}du^2 + \frac{1}{z^2}(-2\,du dv + dx_1^2 + dx_2^2 + dz^2) + ds^2_{X_5}\,, \tag{5.1}$$

where Ω is a function on an internal Einstein manifold, X_5, and n_1 is a real positive constant. The five form flux remains the same, $F_{(5)} = (1+\star)\,\mathrm{vol}_{X_5}$. We complete the Ansatz by introducing a three form flux with the potential

$$B_{(2)} \;=\; \frac{1}{z^{n_2}}\mathcal{A}\wedge du\,, \qquad H_{(3)} \;=\; dB_{(2)}\,, \tag{5.2}$$

where $\mathcal{A}$ is an arbitrary one-form on X_5 and n_2 is a real constant.

The type IIB field equations [35] for this set of fields read

$$R_{MN} \;=\; \frac{1}{6}F_{MPQRS}F_N{}^{PQRS} + \frac{1}{4}H_{MPQ}H_N{}^{PQ}\,, \tag{5.3}$$

[8] For our purposes we will not distinguish between regular and quasi-regular Sasaki-Einstein manifolds [32].

[9] Also called the Reeb vector.

and, taking into account the form of the $H_{(3)}$ flux with nonzero components only along mixed directions,

$$\nabla^M H_{MNP} \;=\; 0\,. \tag{5.4}$$

The only nonvanishing component of the Einstein equations (5.3) is along the uu-direction where it reduces to

$$\frac{1}{z^{2n_1}}\left(\frac{1}{2}\nabla^2_{X_5}\Omega + 2\,(n_1^2+1)\Omega\right) - \frac{4}{z^{2n_1}}\,\Omega \;=\; \frac{1}{z^{2n_2}}\left(\frac{1}{4}\,\mathcal{F}_{\alpha\beta}\mathcal{F}^{\alpha\beta} + \frac{n_2^2}{2}\,\mathcal{A}_\alpha\mathcal{A}^\alpha\right). \tag{5.5}$$

The terms in the bracket on the left hand side arise from the Ricci tensor. The second term comes from the energy momentum tensor of the five-form flux, where the Ω dependence is introduced by the vielbein ${e_u}^M$, see (3.2). Finally, the right hand side arises from the energy momentum tensor of the three-form flux, where $\mathcal{F} = d\mathcal{A}$.

The Maxwell equations (5.4) reduce to two equations. The component of (5.4) along $du \wedge dz$ gives

$$n_2 \nabla^\alpha \mathcal{A}_\alpha \;=\; 0\,. \tag{5.6}$$

The remaining components yield the covariant massive Proca equation

$$\nabla^\alpha \mathcal{F}_{\alpha\beta} + (n_2^2 + 2n_2)\,\mathcal{A}_\beta \;=\; 0\,. \tag{5.7}$$

Expressing components of $\mathcal{F}$ in a covariant form,

$$\mathcal{F}_{\alpha\beta} \;=\; \nabla_\alpha\mathcal{A}_\beta - \nabla_\beta\mathcal{A}_\alpha\,, \tag{5.8}$$

and using the transversality condition (5.6), and $R_{\alpha\beta} = 4g_{\alpha\beta}$, the latter equation can be rewritten as[10]

$$(\nabla^2_{X_5} - 4)\mathcal{A}_\alpha + (n_2^2 + 2n_2)\,\mathcal{A}_\alpha \;=\; 0\,, \tag{5.9}$$

which is the covariant Laplace equation for vector fields on X_5.

Let us first discuss the solutions of (5.5) and (5.9) in the known cases.

- $\mathcal{A}_\alpha = 0$

  We can solve Maxwell equation (5.9) by setting $\mathcal{A}_\alpha = 0$. In this case one is left with a Laplace equation

  $$\nabla^2_{X_5}\Omega + 4(n^2-1)\,\Omega \;=\; 0\,, \tag{5.10}$$

  on the scalar harmonics on the Einstein manifold, X_5, where we set $n = n_1$. This case has been discussed in detail in [15]. Here we only note that the discrete eigenvalues of the Laplacian determine the discrete set of dynamical exponents n. The specific values depend of course on the choice of X_5.

When $\mathcal{A}_\alpha$ does not vanish, the two exponents must be equal, $n_1 = n_2 = n$. Indeed, since both terms on the right hand side in (5.5) are manifestly positive, the powers of z on both sides of the equation must be the same.

[10] The operator $(\nabla^2_{X_5} - 4)$ is, in our normalization, the Lichnerowicz operator on vector fields on the Einstein manifold X_5.

- $\mathcal{A}_\alpha = \mathcal{K}_\alpha$ is a Killing vector
  In this case we can use the standard fact that on an Einstein manifold Killing vectors are eigenfunctions of the Laplacian. Since we choose normalizations such that the internal metric is of unit radius, we have,[11]

$$\nabla^2_{X_5} \mathcal{K}_\alpha \;=\; -\,4\,\mathcal{K}_\alpha\,. \tag{5.11}$$

  This solves (5.9) for $n = 2$. For this value of n, all terms without derivatives in (5.5) cancel if we take $\Omega = \mathcal{K}_\alpha \mathcal{K}^\alpha$. Then the derivative terms combine into

$$\mathcal{K}^\alpha \nabla^2_{X_5} \mathcal{K}_\alpha + \frac{1}{2}\left(\nabla_\alpha \mathcal{K}_\beta + \nabla_\beta \mathcal{K}_\alpha\right) \nabla^\alpha \mathcal{K}^\beta + 4\,\mathcal{K}_\alpha \mathcal{K}^\alpha \;=\; 0\,, \tag{5.12}$$

  which obviously reduces to (5.11). This verifies explicitly that the backgrounds obtained by the null Melvin twist along a Killing vector solve the type IIB equations of motion.

In the general case, we have a coupled system of Laplace equations for the vector field, $\mathcal{A}_\alpha$, and the function, Ω, on X_5,

$$(\nabla^2_{X_5} - 4)\,\mathcal{A}_\alpha \;=\; -\,n(n+2)\,\mathcal{A}_\alpha\,, \tag{5.13}$$

$$\nabla^2_{X_5}\Omega + 4(n^2-1)\,\Omega \;=\; \mathcal{T}(\mathcal{A})\,, \tag{5.14}$$

where

$$\mathcal{T}(\mathcal{A}) \;=\; \frac{1}{2}\,\mathcal{F}_{\alpha\beta}\mathcal{F}^{\alpha\beta} + n^2\,\mathcal{A}_\alpha \mathcal{A}^\alpha\,, \tag{5.15}$$

is a scalar function on X_5.

It is clear that, at least in principle, the system (5.13)–(5.15) can be solved systematically using harmonic analysis on X_5. In the first step one determines the spectrum of the operator $(\nabla^2_{X_5} - 4)$ on vector fields, which in turn determines the values of allowed exponents n in (5.13). For a given eigenvalue of the vector Laplacian there is a degeneracy in the spectrum of vector harmonics. This degeneracy depends on X_5 and will lead to a family of solutions for a fixed eigenvalue. Next, for a given n, one solves (5.13) by setting $\mathcal{A}_\alpha$ to be one of the vector harmonics for the corresponding eigenvalue. The scalar function $\mathcal{T}(\mathcal{A})$ becomes then a source for the inhomogeneous massive Laplace equation (5.14) for the function Ω, which may be solved by expanding $\mathcal{T}(\mathcal{A})$ into scalar spherical harmonics. We will illustrate this procedure below by explicitly working out some solutions for $X_5 = S^5$ and by mapping out the relation between vector eigenvalues of the Laplacian on $T^{1,1}$ and dynamical exponents.

It is worth emphasizing that, for a given vector harmonic $\mathcal{A}_\alpha$, the scalar function Ω in (5.14) may not be unique. This happens when $-4(n^2-1)$, as determined by the eigenvalue of the vector harmonic, $\mathcal{A}_\alpha$, is an eigenvalue of the scalar Laplacian. Then one can add to Ω a solution Ω_0 of the homogenous equation

$$\nabla^2_{X_5}\Omega_0 + 4(n^2-1)\,\Omega_0 = 0\,. \tag{5.16}$$

[11] See, (A.17) in the appendix.

In the next subsection we will see an example of such non-uniqueness in the case when $X_5 = S^5$.

A potential problem with a general solution $(\mathcal{A}_\alpha, \Omega)$ of (5.13)–(5.15) is that there will be regions in X_5 where the function Ω becomes negative. This will change the causal and asymptotic structure of the ten-dimensional solution and may lead to instabilities [15]. Clearly, it would be interesting to understand properties of such solutions in more detail and, in particular, to analyze their role, if any, in non-relativistic holography.

Finally, let us note that some of the solutions with $n \neq 2$ and Sasaki-Einstein manifold X_5 may preserve some supersymmetry. As we discussed in section 3, if the function Ω is non-zero, the superconformal symmetries are broken, but some of the Killing spinors on X_5 may be preserved. In the special case when $\mathcal{A} = 0$ and X_5 is a Sasaki-Einstein manifold, one can use the supersymmetry variations in section 3 to show[12] that the ten-dimensional solutions preserve two supercharges. For $X_5 = S^5$, the number of supercharges increases to eight. We have not performed a general analysis of the Killing spinor equations for $n \neq 2$ and $\mathcal{A} \neq 0$, but it would be interesting if some of them turned out to be supersymmetric.

5.1 Vector harmonics on S^5

The scalar and vector harmonics on spheres were extensively discussed in the literature in the context of the Kaluza-Klein reduction of supergravities.[13] The few basic facts that we need here are derived in [40] and [41], where also earlier references can be found.

All scalar and vector harmonics on S^5 can be constructed from the basic scalar harmonic Y^A and the basic vector harmonic Y^A_α that transform in the vector representation of SO(6) with components labeled by the index $A = 1, \ldots, 6$. They satisfy the following algebraic constraints

$$\sum_{\alpha=1}^{5} Y^A_\alpha Y^B_\alpha \;=\; -\,Y^A Y^B + \delta^{AB}\,, \qquad \sum_{A=1}^{6} Y^A Y^A \;=\; 1\,, \tag{5.17}$$

and form a closed system under covariant differentiation

$$\nabla_\alpha Y^A \;=\; Y^A_\alpha\,, \qquad \nabla_\alpha Y^A_\beta \;=\; -\,\delta_{\alpha\beta} Y^A\,. \tag{5.18}$$

All scalar harmonics are labeled by the totally symmetric traceless representations of SO(6) and are given by

$$Y^{A_1\ldots A_p} \;=\; Y^{(A_1} Y^{A_2} \ldots Y^{A_p)}\,, \qquad p = 0, 1, \ldots\,. \tag{5.19}$$

Similarly, all transverse vector harmonics are of the form

$$Y^{A_1\ldots A_{k+1}}_\alpha \;=\; Y^{[A_1}_\alpha Y^{(A_2]} \ldots Y^{A_{k+1})}\,, \qquad k = 1, 2, \ldots\,, \tag{5.20}$$

where the indices are symmetrized according to the SO(6) hook Young tableaux with k boxes in the first row and one in the second row. There are also longitudinal vector

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[12] This has also been shown in [15].

[13] For a recent comprehensive review in the present context the reader may consult [40]. We thank Peter van Nieuwenhuizen for making this article available to us before publication.

harmonics that are obtained by differentiating the scalar harmonics. We will not need them here since $\mathcal{A}_\alpha$ is a transverse vector harmonic (5.6).

Identities (5.18) turn all covariant differential operators acting on harmonics into algebraic operations, while the constraints (5.17) can be used to reduce products of basic harmonics into irreducible components. In particular, following those steps, one obtains the familiar result for the eigenvalues of the Laplacian used in [42]

$$\nabla^2 Y^{A_1\ldots A_p} \;=\; -\,p(p+4)\,Y^{A_1\ldots A_p}\,, \qquad p=0,1,\ldots\,, \tag{5.21}$$

$$(\nabla^2-4)\,Y_\alpha^{A_1\ldots A_{k+1}} \;=\; -\,(k+1)(k+3)\,Y_\alpha^{A_1\ldots A_{k+1}}\,, \qquad k=1,2,\ldots\,. \tag{5.22}$$

Comparing with (5.13) we have $n=k+1$ and if we define $p=2(n-1)$ it is clear from (5.16) that we can add a homogeneous solution to Ω.[14]

The problem of solving (5.13)–(5.15) is now reduced to a finite dimensional linear algebra. Suppose that we start by choosing $\mathcal{A}_\alpha$ as one of the vector harmonics of order k, which is a polynomial of order $k+1$ in the basic harmonics. Since the differentiation does not increase that order, we conclude that $\mathcal{T}(\mathcal{A})$ is a polynomial of order $2k+2$ or less in the basic *scalar* harmonics. This shows that Ω can be written as a finite sum

$$\Omega \;=\; \sum c_{A_1\ldots A_{2k+2}} Y^{(A_1}Y^{A_2}\ldots Y^{A_{2k+2})}\,, \tag{5.23}$$

where the constant coefficients $c_{A_1\ldots A_{2k+2}}$ are then determined from the scalar equation.

Let us illustrate this using the already familiar case when $\mathcal{A}_\alpha$ is a SO(6) Killing vector. The latter are given by the transverse vector harmonics with $k=1$, which corresponds to $n=2$ in (5.13). First consider

$$\mathcal{A}_\alpha \;=\; Y_\alpha^{AB} \;=\; Y_\alpha^A Y^B - Y_\alpha^B Y^A\,, \tag{5.24}$$

for some fixed A and B. Using (5.17) and (5.18) we obtain

$$\mathcal{A}_\alpha\mathcal{A}^\alpha \;=\; (Y^A)^2+(Y^B)^2 \tag{5.25}$$

$$\;=\; Y^{AA}+Y^{BB}+\frac{1}{3}\,, \tag{5.26}$$

where the constant in the second line arises from subtracting traces in the reduction to SO(6) irreducible components. Similarly,

$$\mathcal{F}_{\alpha\beta} \;=\; -\,2\left(Y_\alpha^A Y_\beta^B - Y_\beta^A Y_\alpha^B\right)\,, \tag{5.27}$$

and

$$\mathcal{F}_{\alpha\beta}\mathcal{F}^{\alpha\beta} \;=\; 8\,\left[1-(Y^A)^2-(Y^B)^2\right]\,. \tag{5.28}$$

Hence

$$\mathcal{T}(\mathcal{A}) \;=\; 4\left[1-(Y^A)^2-(Y^B)^2\right]+4\,\left[(Y^A)^2+(Y^B)^2\right] \;=\; 4\,. \tag{5.29}$$

[14]The freedom to add homogeneous solutions to Ω for any value of n is specific to S^5 and will not be present for a general Einstein manifold.

More generally, we take a linear combination of such harmonics,

$$\mathcal{A}_\alpha \;=\; \frac{1}{2}\sum_{A,B}\eta_{AB}\,Y_\alpha^{AB} \;=\; \sum_{A,B}\eta_{AB}\,Y_\alpha^A Y^B\,, \tag{5.30}$$

where $\eta_{AB} = -\eta_{BA}$. Then

$$\mathcal{A}_\alpha\mathcal{A}^\alpha \;=\; \sum_{A,B,C}\eta_{AC}\eta_{BC}Y^{AB} + \frac{1}{3}\eta^2\,, \qquad \mathcal{T}(\mathcal{A}) \;=\; 4\eta^2\,, \tag{5.31}$$

where

$$\eta^2 \;=\; \sum_{A<B}\eta_{AB}^2 \qquad \text{and} \qquad Y^{AB} = Y^A Y^B - \frac{1}{6}\delta^{AB}\,. \tag{5.32}$$

Substituting this in the scalar equation (5.14) with $n = 2$, we get

$$\nabla^2\,\Omega + 12\,\Omega \;=\; 4\eta^2\,. \tag{5.33}$$

The generic solution to this equation, which we discussed above, is $\Omega = \mathcal{A}_\alpha\mathcal{A}^\alpha$. However, there is another obvious solution, which is simply the constant function

$$\Omega \;=\; \frac{1}{3}\eta^2\,. \tag{5.34}$$

From (5.31), the difference between the two solutions is a sum of $k = 2$ scalar harmonics, $Y^{A_1A_2}$, which satisfy the homogenous equation (5.16) (cf. (5.21)).

To summarize, we have shown that there is a $15 + 20$ parameter family[15] of solutions on S^5

$$\mathcal{A}_\alpha \;=\; \mathcal{K}_\alpha\,, \qquad \Omega \;=\; ||\mathcal{K}||^2 + Y\,, \tag{5.35}$$

where $\mathcal{K}_\alpha$ is a Killing vector and Y is a solution to (5.14) with $k = 2$.

The solutions with $\Omega = \mathcal{K}_\alpha\mathcal{K}^\alpha$ and $Y = 0$ arise naturally from the null Melvin twist construction. If one takes $\mathcal{K}_\alpha$ to be a Killing vector along the Hopf fiber, its length is constant and the solution reduces to (72) with constant Ω. However, since all Killing vectors on S^5 are equivalent under the SO(6) symmetry, the solution with constant Ω should exist for any choice of $\mathcal{K}_\alpha$, which indeed is the case.

As a final example we consider a solution with a higher vector harmonic. Let us take the harmonic [40]

$$Y_\alpha^{ABC} \;=\; 2Y_\alpha^A Y^B Y^C - Y_\alpha^B Y^A Y^C - Y_\alpha^C Y^A Y^B\,, \tag{5.36}$$

with distinct A, B and C. It has $k = 2$, which gives the dynamical exponent $n = 3$. Similarly as above, we find

$$\mathcal{T}(\mathcal{A}) \;=\; 9\left[(Y^B)^2 + (Y^C)^2\right]\,. \tag{5.37}$$

[15] There is a 15 parameter degeneracy for the vector harmonic with $k = 1$ and a 20 parameter degeneracy for the scalar harmonic with $p = 2$. It is possible that some of these solutions are equivalent.

However, unlike before, the mass term in (5.16) is equal to 32, which does not correspond to any of the eigenvalues of the scalar Laplacian in (5.21). Hence the solution

$$\Omega \;=\; \frac{9}{20}\left[(Y^B)^2 + (Y^C)^2 - \frac{1}{8}\right] , \tag{5.38}$$

is unique, up to the degeneracy in the choice of vector harmonic and addition of a homogeneous solution discussed above. This example illustrates explicitly the problem we mentioned earlier that, for higher order harmonics, Ω may be negative in some region of X_5.

We conclude the discussion of S^5 with the observation that the eigenvalues for the transverse vector harmonics in (5.22) determine the dynamical exponents of our solutions to take values

$$n = k + 1 \,. \tag{5.39}$$

Hence, we have here examples of scale-invariant, non-relativistic type IIB solutions with a B-field and integer dynamical exponents, $n \geq 2$.

5.2 Vector harmonics on $T^{1,1}$

We will not attempt to find explicit solutions with different dynamical exponents generated by the vector harmonics on $T^{1,1}$. Instead we will map out the relation between the spectroscopy of vector eigenfunctions of the Laplacian on $T^{1,1}$ and the dynamical exponents of the field theories dual to the new gravity solutions.

The eigenfunctions of the vector Laplacian are labeled by weights of the $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ isometry group of $T^{1,1}$ [43, 44],

$$\nabla^2 \mathcal{A}_\alpha^{l_1,l_2,l_3} = \lambda(l_1,l_2,l_3)\mathcal{A}_\alpha^{l_1,l_2,l_3} \,. \tag{5.40}$$

There are four series of eigenvalues

$$\lambda_{1,2} \;=\; 3 + h(l_1,l_2,l_3 \pm 2)\,, \tag{5.41}$$

$$\lambda_{3,4} \;=\; h + 4 \pm 2\sqrt{h+4}\,, \tag{5.42}$$

where $h(l_1,l_2,l_3)$ are the scalar eigenvalues of the Laplacian on $T^{1,1}$

$$h(l_1,l_2,l_3) \;=\; 6\left(l_1(l_1+1) + l_2(l_2+1) - \frac{l_3^2}{8}\right) . \tag{5.43}$$

Here l_1, l_2 could be integers or half-integers and l_3 is an integer. The values of the dynamical exponents of the gravitational backgrounds are determined by the solutions to any of the four algebraic equations

$$n(n+2) \;=\; 4 - \lambda_i(l_1,l_2,l_3)\,, \qquad\qquad i = 1,2,3,4\,. \tag{5.44}$$

It follows that for $T^{1,1}$ as the internal manifold, the dynamical exponents of the non-relativistic solutions are not arbitrary integers, as is the case for S^5. In fact, for generic values of (l_1,l_2,l_3) the dynamical exponent, n, is irrational.

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6 Conclusions

We have found a large class of supersymmetric IIB solutions with non-vanishing B-field which are invariant under the Schrödinger symmetry. The solutions are obtained by the null Melvin twist from supersymmetric type IIB solutions of the form $AdS_5 \times X_5$. The field theories dual to these solutions form a very special class of non-relativistic field theories. They can be obtained from the relativistic $\mathcal{N} = 1$ superconformal Yang-Mills theories dual to $AdS_5 \times X_5$ by performing a discrete light-cone quantization accompanied by a twist. The twist amounts to modifying all products of chiral superfields in the Lagrangian of the relativistic theory by phases proportional to the charges of the fields under the U(1) global symmetries used in the null Melvin twist. This class of field theories was discussed in [45–47], see also [11].

We also found a quite general class of type IIB solutions with dynamical exponents different from two and non-vanishing B-field. As discussed in section 5, the B-field is determined by a vector harmonic on an Einstein manifold, X_5. The metric is obtained by solving an inhomogeneous scalar Laplace equation on X_5. The solutions are invariant under the Galilean group and dilatations, but are not invariant under special conformal transformations, and thus break the full Schrödinger symmetry. The dual field theories should be scale invariant and invariant under Galilean transformations. Since we did not generate the gravity solutions by some twist of known relativistic solutions with clear D-brane interpretation, the detailed structure of the dual non-relativistic field theories is not clear at present. It would be very interesting to reduce the solutions of section 5 to five dimensions and to see whether they can be obtained as solutions to five-dimensional gravity coupled to some massive fields [7, 8, 10]. More generally, it is important to understand compactifications and consistent truncations of type IIB supergravity with massive fields [10, 48].

There are several directions in which our analysis could be extended. It is interesting to find eleven-dimensional analogs of our solutions. Some supersymmetric non-relativistic solutions of the eleven-dimensional supergravity are known, [19, 21]. Perhaps it is possible to find more general classes of such solutions using the results of this paper as a guide. A natural way to proceed is to start with a Freund-Rubin solution of the form $AdS_4 \times X_7$ where X_7 is a seven-dimensional Sasaki-Einstein manifold with $\mathrm{U}(1)^4$ isometry.[16] Then one can reduce the solution along the U(1) R-symmetry to get a solution of IIA supergravity with $\mathrm{U}(1)^3$ symmetry. The generalized null Melvin twist applied to this solution would generate a non-relativistic type IIA solution with a Sch_4 non-compact space-time. This solution could be uplifted to a solution of eleven-dimensional supergravity. It is natural to expect that for the null Melvin twist performed along U(1) isometries that leave the X_7 Killing spinor invariant, the twisted eleven-dimensional background will preserve some supersymmetry. It should also be possible to use this eleven-dimensional solution as a guide for constructing more general solutions along the lines of section 5.

There are supersymmetric extensions of the Schrödinger algebra with various amounts of supersymmetry [31]. It is interesting to explore the connection between these superalge-

[16]There exists an infinite family of such manifolds with explicitly known metrics [33, 49].

bras and the supersymmetric Schrödinger invariant IIB solutions found here and also those in [15, 19]. It is natural to explore whether any of the supersymmetric type IIB solutions presented here could be realized as non-relativistic supercosets along the lines of [50].

We would like to note that it is straightforward to find finite temperature counterparts of all solutions discussed in section 2 and section 4. One should start with a $AdS_5 \times X_5$ black hole solution and apply the generalized null Melvin twist. This was done in [22] for the $AdS_5 \times S^5$ black hole. It would be also interesting to construct the finite temperature versions of the solutions with general dynamical exponents found in section 5 and explore their thermodynamics.

Finally it is tempting to speculate that there might be supersymmetric (and non-supersymmetric) non-relativistic analogs of the familiar RG flow solutions of the type IIB and eleven-dimensional supergravities. A modest attempt to construct such solutions was made in [22], but there is certainly much more to explore.

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A Generalized null Melvin twist

The null Melvin twist [24, 25] (see, also [9, 11, 22]) is a solution generating technique which can be used to construct explicitly new solutions of the type IIB supergravity starting from a known solution with at least U(1) isometry. In this appendix we summarize the main steps of the construction as applied recently in [22] to solutions with $\mathrm{U}(1)^3$ isometry. We show that by examining explicit formulae for the twisted solutions in [22], one is naturally led to rewrite the result of the null Melvin twist along any isometry given by a Killing vector, $\mathcal{K}$, in terms of intrinsic quantities without reference to any specific coordinates.

Consider a type IIB solution of the form $AdS_5 \times X_5$, where X_5 is an Einstein manifold with $\mathrm{U}(1)^3$ isometry. For definiteness, let us take

$$ds^2_{10} = \frac{1}{z^2}(-dt^2 + dy^2 + dx_1^2 + dx_2^2 + dz^2) + ds^2_{X_5}\,, \tag{A.1}$$

$$F_{(5)} = (1 + \star)\ \mathrm{vol}_{AdS_5}\,, \tag{A.2}$$

where we have normalized the AdS_5 and X_5 to be of unit radius. In all cases of interest, the metric on the internal manifold X_5 can be written, at least locally, in the form

$$ds^2_{X_5} = f_1 d\theta_1^2 + f_2 d\theta_2^2 + \sum_{i,j=1}^{3} f_{ij}\, d\phi_i d\phi_j\,, \tag{A.3}$$

where all functions f_i and f_{ij} depend only on θ_1 and θ_2. The angles ϕ_1, ϕ_2, ϕ_3 parametrize directions along the three U(1)'s.

The generalized null Melvin twist [22, 24, 25] consists of the following operations that are straightforward to implement:

- a boost in the (t, y) plane with parameter γ_0,
- a T-duality along y,
- a shift of all three U(1) isometries of X^5 given by $\phi_i \to \phi_i + a_i y$,
- another T-duality along y,
- an inverse boost in the (t, y) plane with parameter $-\gamma_0$,
- a limit $a_i \to 0$, $\gamma_0 \to \infty$ such that $\eta_i \equiv a_i \cosh \gamma_0 = a_i \sinh \gamma_0$ remain finite.

Introducing light-cone coordinates

$$u = t + y\,, \qquad v = \frac{1}{2}(t - y)\,, \tag{A.4}$$

the resulting spacetime is a product space $Sch_5 \times X_5$, with the metric

$$ds^2_{10} = -\frac{\Omega}{z^4}\, du^2 + \frac{1}{z^2}(-2du dv + dx_1^2 + dx_2^2 + dz^2) + ds^2_{X_5}\,, \tag{A.5}$$

a nontrivial two-form potential

$$B_{(2)} = \frac{1}{2z^2}[\partial_{\eta_1}\Omega\; d\phi_1 + \partial_{\eta_2}\Omega\; d\phi_2 + \partial_{\eta_3}\Omega\; d\phi_3] \wedge du\,, \tag{A.6}$$

and the five-form flux

$$F_{(5)} = (1 + \star)\mathrm{vol}_{Sch_5}\,, \tag{A.7}$$

where Ω is explicitly given by

$$\Omega(\theta_1, \theta_2) = \sum_{i,j=1}^{3} f_{ij}\, \eta_i \eta_j\,. \tag{A.8}$$

Note that since $\mathrm{vol}_{AdS_5} = \mathrm{vol}_{Sch_5}$, the five-form flux is in fact invariant under the twist.

The entire twisted solution is completely determined by Ω in (A.8). In order that the metric in (A.5) is well defined over the entire space, $Sch_5 \times X_5$, this Ω must be a scalar function on X_5. This in turn implies that the twist parameters η_i's should be viewed as coordinates of a vector field on X_5, rather than constants. Indeed, the solution (A.5)–(A.7) can be easily recast into a form that makes this manifest.

The Killing vectors of the $\mathrm{U}(1)^3$ isometry for the internal metric (A.3) are linear combinations of the three Killing vectors

$$\mathcal{K}_{(i)} = \frac{\partial}{\partial \phi_i}\,, \qquad i = 1, 2, 3\,. \tag{A.9}$$

We observe that in terms of the Killing vector

$$\mathcal{K} \;=\; \sum_{i=1}^{3} \eta_i\, \mathcal{K}_{(i)} \;=\; \sum_{i=1}^{3} \eta_i\, \frac{\partial}{\partial \phi_i}\,, \tag{A.10}$$

we have simply

$$\Omega = ||\mathcal{K}||^2\,, \tag{A.11}$$

where $||\mathcal{K}||$ is the norm of $\mathcal{K}$. Furthermore,

$$B_{(2)} \;=\; \frac{1}{z^2}\,\mathcal{K}\wedge du\,, \tag{A.12}$$

where $\mathcal{K}$ is the one form dual to Killing vector, $\mathcal{K}$, with respect to the internal metric.

Eqs. (A.11) and (A.12) show that the twisted solution is well defined over the entire internal manifold. In particular, one can use them to write down the solution (A.5)–(A.8) in terms of arbitrary coordinates (ξ^α) on X_5,

$$ds^2_{X_5} \;=\; g_{\alpha\beta}\, d\xi^\alpha d\xi^\beta\,, \qquad R_{\alpha\beta} \;=\; 4\, g_{\alpha\beta}\,, \tag{A.13}$$

and

$$\Omega = g_{\alpha\beta}\,\mathcal{K}^\alpha\mathcal{K}^\beta\,, \qquad \mathcal{K}_\alpha = g_{\alpha\beta}\,\mathcal{K}^\beta\,. \tag{A.14}$$

Since the metric is block diagonal, (A.14) has the same form when written in terms of ten-dimensional coordinates.

The explicit background fields in (A.5)–(A.8) were obtained by applying the null Melvin twist along an arbitrary Killing vector. Therefore, by construction one is guaranteed to get a solution of the type IIB supergravity. However, it is also illuminating to verify this explicitly starting with an Ansatz for the fields as in (A.5)–(A.7), (A.11) and (A.12), where X_5 is an arbitrary Einstein manifold with a globally defined vector field $\mathcal{K}$. Using the formulae for the spin connection and the fluxes in section 3, we find that the Maxwell and the Einstein equations reduce to two equations for $\mathcal{K}$,

$$\nabla^2_{X_5}\,\mathcal{K}_\alpha + 4\,\mathcal{K}_\alpha \;=\; 0\,, \tag{A.15}$$

and

$$\mathcal{K}^\alpha\nabla^2_{X_5}\,\mathcal{K}_\alpha + \frac{1}{2}\left(\nabla_\alpha\mathcal{K}_\beta + \nabla_\beta\mathcal{K}_\alpha\right)\nabla^\alpha\mathcal{K}^\beta + 4\,\mathcal{K}_\alpha\mathcal{K}^\alpha \;=\; 0\,, \tag{A.16}$$

respectively, where $\nabla^2_{X_5} = \nabla^\alpha\nabla_\alpha$ is the covariant Laplacian on X_5. The two equations imply that $\nabla_{(\alpha}\mathcal{K}_{\beta)} = 0$, as the second term in (A.16) is manifestly positive. Thus $\mathcal{K}^\alpha$ must be a Killing vector and (A.16) follows from (A.15). It is a standard fact that Killing vectors are eigenfunctions of the Laplacian on an Einstein manifold so that the latter equation is always satisfied. Indeed, we have

$$\begin{aligned} \nabla^\alpha(\nabla_\alpha\mathcal{K}_\beta + \nabla_\beta\mathcal{K}_\alpha) \;&=\; \nabla^2_{X_5}\,\mathcal{K}_\beta + R^\alpha{}_\beta\mathcal{K}_\alpha \\ &=\; \nabla^2_{X_5}\,\mathcal{K}_\beta + 4\,\mathcal{K}_\beta\,, \end{aligned} \tag{A.17}$$

where we used that $\nabla^\alpha\mathcal{K}_\alpha = 0$. Note that the normalization of the mass term in (A.15) corresponds to the unit radius of X_5 in (A.13).

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